Indian Statistical Institute, Bangalore Centre
M.Math I Year, Second Semester
Solution set of Mid-Sem Examination 2012
Functional Analysis

1. Let $(\mathcal{H}, \langle , \rangle)$ be a Hilbert space. A map $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is said to be sesquilinear iff it is linear in the first variable and conjugate linear in the second variable. Define

$$\|B\| := \sup_{\|x\|\leq 1, \|y\|\leq 1} |B(x, y)|.$$

Say that B is bounded if $\|B\| < \infty$; is symmetric if $B(x, y) = \overline{B(y, x)}$; is non-negative if $B(x, x) = 0$ implies $x = 0$.

(a) If B is bounded, show that there exists a unique bounded linear operator A on H such that $B(x, y) = \langle x, Ay\rangle$ and that $\|B\| = \|A\|$.

(b) Verify that if B is symmetric, non-negative and definite then B defines an inner product on H.

(c) If B is symmetric and non-negative, show that the Schwartz inequality holds: for every $x, y \in H$,

$$|B(x, y)|^2 \leq B(x, x)B(y, y).$$

Proof. (a) For each $y \in \mathcal{H}$, define the map $B_y : \mathcal{H} \to \mathbb{C}$ by

$$B_y(x) = B(x, y)$$

for each $x \in \mathcal{H}$. Since $B(\alpha x_1 + x_2, y) = \alpha B(x_1, y) + B(x_2, y)$, B_y is a linear map. From the definition of $\|B\|$, we can write for $x, y \in \mathcal{H}$

$$|B(x, y)| \leq \|B\|\|x\|\|y\|$$

which implies that B_y is bounded and $\|B_y\| \leq \|B\|\|y\|$. Since B_y is a bounded linear functional on the Hilbert space $\mathcal{H}$, there exists a unique $\tilde{y} \in \mathcal{H}$ such that $(B(x, y) =)B_y(x) = \langle x, \tilde{y}\rangle$ and $\|B_y\| = \|\tilde{y}\|$. Because for each $y \in \mathcal{H}$ we are able to associate a unique $\tilde{y} \in \mathcal{H}$ we can define a map $A : \mathcal{H} \to \mathcal{H}$ by the relation $Ay = \tilde{y}$. Since B is sesquilinear, $A : \mathcal{H} \to \mathcal{H}$ is a linear map. Further $(\|\tilde{y}\| =)\|Ay\| = \|B_y\| \leq \|B\|\|y\|$ implies A is bounded and $\|A\| \leq \|B\|$. For each $x, y \in \mathcal{H}$

$$|B(x, y)| = |\langle x, Ay\rangle| \leq \|x\|\|Ay\|$$

implies that $\|B\| \leq \|A\|$. Thus $\|B\| = \|A\|$.

(b) Define $\langle x, y\rangle := B(x, y)$. Since B is non-negative and definite, $\langle x, x\rangle \geq 0$ and equality holds iff $x = 0$. From the sesquilinearity of B the inner product $\langle , \rangle$ is linear in the first variable. Further, $\langle x, y\rangle = \overline{\langle y, x\rangle}$ holds for $x, y \in \mathcal{H}$ because B is symmetric. Hence B defines an inner product on $\mathcal{H}$.

(c) Let $x, y \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Then using properties of B, we have

$$0 \leq B(x - \lambda y, x - \lambda y) = B(x, x) - 2Re(\bar{\lambda} B(x, y)) + |\lambda|^2 B(y, y).$$

In particular, set $\lambda = \dfrac{B(x, y)}{\alpha}$ for $\alpha > 0$ so that

$$0 \leq B(x, x) - \frac{1}{\alpha}\big(2 - \frac{B(y, y)}{\alpha}\big)|B(x, y)|^2.$$

If $B(y, y) \neq 0$, then take $\alpha = B(y, y)$ to get the Schwartz inequality. If $B(y, y) = 0$, then

$$0 \leq 2|B(x, y)|^2 \leq \alpha B(x, x)$$

for all $\alpha > 0$, and which forces $B(x, y) = 0$ (in this case the Schwartz inequality holds trivially). $\square$

2. Let M_n be the space of $n \times n$ matrices over $\mathbb{C}$, considered as bounded linear operators on $\mathbb{C}^n$. Let $GL_n(\mathbb{C})$ be the group of $n \times n$ invertible matrices and let U be an open subset of $GL_n(\mathbb{C}) \subset M_n$. Define $J : U \to M_n$ by $J(A) := A^{-1}$. Show that J is Frechet differentiable at all $A \in U$ and that if $H \in M_n$ then $J'(A)H = -A^{-1}HA^{-1}$.

Proof. The map $J : U \to M_n$ is said to be Frechet differentiable at $A \in U$, if there exists a bounded linear operator $J'(A) : M_n \to M_n$ such that

$$\lim_{\|H\| \to 0} \frac{\|J(A + H) - J(A) - J'(A)(H)\|_{M_n}}{\|H\|_{M_n}} = 0.$$

Let $H \in M_n$ be such that $\|H\|$ suffieciently small and $\|A^{-1}\|\|H\| < 1$ (which implies $\|A^{-1}H\| < 1$). Then $I + A^{-1}H$ is invertible and we use this fact in the following expression:

$$\begin{aligned} J(A+H) = (A+H)^{-1} = (A(I+A^{-1}H))^{-1} &= (I+A^{-1}H)^{-1}A^{-1} \\ &= \sum_{k=0}^{\infty}(-1)^k(A^{-1}H)^k A^{-1} \\ &= A^{-1} - A^{-1}HA^{-1} + \sum_{k=2}^{\infty}(-1)^k(A^{-1}H)^k A^{-1} \\ &= J(A) - A^{-1}HA^{-1} + \sum_{k=2}^{\infty}(-1)^k(A^{-1}H)^k A^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} \|J(A+H) - J(A) - (-A^{-1}HA^{-1})\| &= \|\sum_{k=2}^{\infty}(-1)^k(A^{-1}H)^k A^{-1}\| \\ &\leq \sum_{k=2}^{\infty}\|(-1)^k(A^{-1}H)^k A^{-1}\| \\ &\leq [\|A^{-1}\|^2\|H\|^2\|A^{-1}\|]\sum_{k=0}^{\infty}[\|A^{-1}\|\|H\|]^k \\ &\leq [\|A^{-1}\|^2\|H\|^2\|A^{-1}\|]\left(\frac{1}{1-\|A^{-1}\|\|H\|}\right). \end{aligned}$$

If we take $\|H\| \to 0$ both sides in the above inequality, then

$$\lim_{\|H\|\to 0}\|J(A+H) - J(A) - (-A^{-1}HA^{-1})\| = 0.$$

Hence we have proved the required result. □

3. Let $(V, \|.\|)$ be a Banach space and $X = C([0,1], V)$ the space of continuous functions from $[0,1]$ with values in V. For $f \in X$, define $\|f\|_X := \sup\limits_{t\in[0,1]}\|f(t)\|$. Show that $\|.\|_X$ is well defined and is a norm on X. Show that $(X, \|.\|_X)$ is a Banach space.

Proof. First we prove that $\|.\|$ is well-defined. Let $f \in X$. Since the image of a compact set under a contiuous map is compact and a compact subset of any norm linear space is bounded (in fact totally bounded), $\sup_{t\in[0,1]}\|f(t)\|$ is a finite number

for $f \in X$. Thus $\|.\|_X$ is well defined.
Let $f, g, h \in X$ and $\alpha \in \mathbb{C}$.

1. Clearly $\|f\|_X \geq 0$ and if $\|f\|_X = 0$, then $f(t) = 0$ for each $t \in [0,1]$, i.e., $f = 0$.
2. $$\begin{aligned}\|\alpha f\|_X &= \sup_{t\in[0,1]} \|(\alpha f)(t)\| = \sup_{t\in[0,1]} \|(\alpha(f(t))\| = \sup_{t\in[0,1]} |\alpha|\|f(t)\| \\ &= |\alpha| \sup_{t\in[0,1]} \|f(t)\| = |\alpha|\|f\|_X.\end{aligned}$$
3. $$\begin{aligned}\|f+g\|_X &= \sup_{t\in[0,1]} \|(f+g)(t)\| \\ &= \sup_{t\in[0,1]} \|(f(t)+g(t)\| \\ &\leq \sup_{t\in[0,1]} [\|(f(t)+h(t)\| + \|(h(t)+g(t)\|] \quad (\because V \text{ is a normed linear space.}) \\ &\leq \sup_{t\in[0,1]} \|(f(t)+h(t)\| + \sup_{t\in[0,1]} \|(h(t)+g(t)\| \\ &= \|f+h\|_X + \|h+g\|_X.\end{aligned}$$

Let $\{f_n\}$ be a Cauchy sequence in X. Then for each $\epsilon > 0$ there is a positive integer N such that

$$\|f_n(t) - f_m(t)\| \leq \|f_n - f_m\|_X \leq \epsilon \tag{1}$$

for all $n, m \geq N, t \in [0,1]$. Thus $\{f_n(t)\}$ is a Cauchy sequence in the Banach space V. So it is convergent for each $t \in [0,1]$. Define $f(t) = \lim_{n\to\infty} f_n(t)$ for $t \in [0,1]$. Taking $m \to \infty$ in equation (1) we have, for all $n \geq N$

$$\sup_{t\in[0,1]} \|f_n(t) - f(t)\| \leq \epsilon \tag{2}$$

Our next aim to show $f \in X$. Let $t_0 \in [0,1]$. Since f_N is contiuous at t_0, for $\epsilon > 0$ (same as above) there is a $\delta > 0$ such that

$$\|f_N(t) - f_N(t_0)\| < \epsilon \text{ whenever } |t - t_0| < \delta. \tag{3}$$

Now using equations (2) and (3) we get the following:

$$\|f(t) - f(t_0)\| \leq \|f(t) - f_N(t)\| + \|f_N(t) - f_N(t_0)\| + \|f_N(t_0) - f(t_0)\| < 3\varepsilon \quad (4)$$

whenever $|t - t_0| < \delta$. Thus f is a continuous function on $[0, 1]$ as $t_0 \in [0, 1]$ arbitrary. So $f \in X$ and from equation (2) we can write $\|f_n - f\|_X \leq \epsilon$ for all $n \geq N$. Hence we have proved that $(X, \|.\|_X)$ is a Banach space. □

4. Let V be a normed linear space and V^* its dual. For any subspace $Z \subset V^*$ and $f \in V^*$ define $d(f, Z) := \inf_{g \in Z} \|f - g\|_{V^*}$. Let $W \subset V$ be a subspace and define $W^\perp := \{g \in V^* : g(x) = 0, \forall x \in W\}$.

   (a) Show that $W^\perp$ is a closed subspace of V^*.

   (b) Let $f \in V^*$. Show that $d(f, W^\perp) = \|f_0\|_{W^*}$, where $f_0 \in W^*$ is the restriction of $f \in V^*$ to W.

Proof. (a) Let $g_1, g_2 \in W^\perp$ and $\alpha \in \mathbb{C}$. Then for all $x \in W$, we have

$$(\alpha g_1 + g_2)(x) = \alpha g_1(x) + g_2(x) = 0.$$

Thus $\alpha g_1 + g_2 \in W^\perp$. This shows that $W^\perp$ is a subspace of V^*.
Let $\{g_n\}$ be a sequence in $W^\perp$ and $g_n \to f$ in V^*. Then

$$|g_n(x) - f(x)| = |(g_n - f)(x)| \leq \|g_n - f\|\|x\| \to 0$$

as $n \to \infty$ for all $x \in V$. If $x \in W^\perp$, then $g_n(x) = 0$ for all n; hence $f(x) = 0$. Thus $f \in W^\perp$. It means that $W^\perp$ is closed in V^*.

(b) Let $f \in V^*$. Define $f_0 = f|_W$ as the restriction of f to W. Since $g(x) = 0$ for all $g \in W^\perp$,

$$f_0 = (f - g)|_W \quad \text{for all } g \in W^\perp.$$

Then $\|f_0\|_{W^*} = \|(f - g)|_W\|_{W^*} \leq \|f - g\|_{V^*}$ for all $g \in W^\perp$. Thus

$$\|f_0\|_{W^*} \leq \inf_{g \in W^\perp} \|f - g\|_{V^*}. \quad (5)$$

Next we prove the above inequality in other way. By the Hahn-Banach extension theorem there exists an element $h \in V^*$ such that $f_0 = h|_W$ and $\|h\|_{V^*} = \|f_0\|_{W^*}$.

Note that for $x \in W$ we have $(f-h)(x) = f(x) - h(x) = f_0(x) - f_0(x) = 0$. Then $f - h \in W^{\perp}$. Using this fact we get the following:

$$\inf_{g \in W^{\perp}} \|f - g\|_{V^*} \leq \|f - (f-h)\|_{V^*} = \|h\|_{V^*} = \|f_0\|_{W^*}. \tag{6}$$

Hence required result follows from (5) and (6), namely

$$d(f, W^{\perp}) = \|f_0\|_{W^*}.$$

□

5. Let V be a normed linear space. Show that finite dimesional subspaces of V can be complemented, i.e., if $W \subset V$ is finite dimensional, then there exists a closed subspace $Z \subset V$ such that $V = W \oplus Z$.

Proof. Assume that $\dim W = n$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of W. Then each $x \in W$ we can write as

$$x = \alpha_1(x)e_1 + \alpha_2(x)e_2 + \ldots + \alpha_n(x)e_n$$

where $\alpha_j(x)$'s are scalars depend on $x \in W$. Using these scalars, we define linear functional $\alpha_j : W \to \mathbb{C}$ for each $j = 1, \ldots, n$. Since W is finite dimensional, α_j is bounded for each $j = 1, \ldots, n$. Observe that $\alpha_j(e_i) = \delta_{ij}$ for each $i, j = 1, \ldots, n$. By Hahn-Banach extension theorem there exists an extension $f_j \in V^*$ of $\alpha_j \in W^*$ for each $j = 1, \ldots, n$. Define the map $P : V \to V$, for $x \in V$

$$P(x) = f_1(x)e_1 + f_2(x)e_2 + \ldots + f_n(x)e_n$$

Since f_j is a linear map for each $j = 1, \ldots, n$, P is also a linear map. For $x \in V$, we have

$$\begin{aligned}
\|P(x)\| &= \|f_1(x)e_1 + f_2(x)e_2 + \ldots + f_n(x)e_n\| \\
&\leq |f_1(x)|\|e_1\| + |f_2(x)|\|e_2\| + \ldots + |f_n(x)|\|e_n\| \\
&\leq \|f_1\|\|x\|\|e_1\| + \|f_2\|\|x\|\|e_2\| + \ldots + \|f_n\|\|x\|\|e_n\| \\
&\leq [\|f_1\| + \|f_2\| + \ldots + \|f_n\|]\big(\max_{1 \leq j \leq n} \|e_j\|\big)\|x\|\| = K\|x\|
\end{aligned}$$

where $K = [\|f_1\|+\|f_2\|+\ldots+\|f_n\|] \max_{1\leq j\leq n} \|e_j\|$. So P is a bounded linear operator. Note that

$$\begin{aligned} P(e_j) &= f_1(e_j)e_1 + f_2(e_j)e_2 + \ldots + f_n(e_j)e_n \\ &= \alpha_1(e_j)e_1 + \alpha_2(e_j)e_2 + \ldots + \alpha_n(e_j)e_n = e_j \end{aligned}$$

where in the last equality we have used the fact $\alpha_j(e_i) = \delta_{ij}$. We claim that P is the projection on to W. For that we need to show $P^2 = P$ and $\mathrm{Ran}P = W$. Let $x \in V$.

$$\begin{aligned} P^2(x) &= P(f_1(x)e_1 + f_2(x)e_2 + \ldots + f_n(x)e_n) \\ &= f_1(x)P(e_1) + f_2(x)P(e_2) + \ldots + f_n(x)P(e_n) \\ &= f_1(x)e_1 + f_2(x)e_2 + \ldots + f_n(x)e_n = P(x) \end{aligned}$$

Thus $P^2 = P$. By the definition of P it is clear that $\mathrm{Ran}P = W$. So P is the projection on to W. Take $Z = \ker P (= \mathrm{Ran}(I-P))$. Then $W \cap Z = \{0\}$. For any $x \in V$ is written as $x = Px + (I-P)x \in W + Z$. Hence $V = W \oplus Z$. $\square$